Malaysian Journal of Mathematical Sciences 8(1): 47-68 (2014)

MALAYSIAN JOURNAL OF MATHEMATICAL SCIENCES

Journal homepage: http://einspem.upm.edu.my/journal

# On Friendly Index Sets of Spiders

**[1]Sin-Min Lee, [2]Ho-Kuen Ng and [3*]Gee-Choon Lau**

*[1]34803, Hollyhock Street,Union City, CA 94587 USA*

*[2] Department of Mathematics, San Jose State University, San Jose, CA 95192 USA*

*[3]Faculty of Computer & Mathematical Sciences, Universiti Teknologi MARA (Segamat Campus), 85000 Johor, Malaysia*

*E-mail: geeclau@yahoo.com*

*Corresponding author

## ABSTRACT

Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$, and let $A$ be an abelian group. A labeling $f : V(G) \to A$ induces an edge labeling $f^* : E(G) \to A$ defined by $f^*(xy) = f(x)+f(y)$, for each edge $xy \in E(G)$. For $i \in A$, let $v_f(i) = |\{v \in V(G) : f(v) = i\}|$ and $e_f(i) = |\{e \in E(G) : f^*(e) = i\}|$. Let $c(f) = \{|e_f(i) - e_f(j)| : (i, j) \in A \times A\}$. A labeling $f$ of a graph $G$ is said to be $A$-friendly if $|v_f(i) - v_f(j)| \le 1$ for all $(i, j) \in A \times A$. If $c(f)$ is a (0, 1)-matrix for an $A$-friendly labeling $f$, then $f$ is said to be $A$-cordial. When $A = Z_2$, the friendly index set of the graph $G$, $FI(G)$, is defined as $\{| e_f(0) - e_f(0)|$: the vertex labeling $f$ is $Z_2$-friendly$\}$. In this paper, we determined the friendly index sets of many spiders.

Keywords: Vertex labelling, friendly labelling, cordiality, spider, tree.

## 1. INTRODUCTION

Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$, and let $A$ be an abelian group. A labeling $f : V(G) \to A$ induces an edge labeling $f^* : E(G) \to A$ defined by $f^*(xy) = f(x) + f(y)$, for each edge $xy \in E(G)$. For $i \in A$, let $v_i(i) = |\{v \in V(G) : f(v) = i\}|$ and $e_f(i) = |\{e \in E(G) : f^*(e) = i\}|$. Let $c(f) = \{|e_f(i) - e_f(j)| : (i, j) \in A \times A\}$. A labeling $f$ of a graph $G$ is said to be $A$-

friendly if $|v_f(i) - v_f(j)| \leq 1$ for all $(i, j) \in A \times A$. If $c(f)$ is a (0, 1)-matrix for an $A$-friendly labeling $f$, then $f$ is said to be $A$-cordial.

The notion of $A$-cordial labelings was first introduced by Hovay in 1991, who generalized the concept of cordial graphs of Cahit in 1987. Cahit considered $A = Z_2$ and he proved the following: every tree is cordial; $K_n$ is cordial if and only if $n \leq 3$; $K_{m,n}$ is cordial for all $m$ and $n$; the wheel $W_n$ is cordial if and only if $n \not\equiv 3 \pmod 4$; $C_n$ is cordial if and only if $n \not\equiv 2 \pmod 4$; and an Eulerian graph is not cordial if its size is congruent to 2 (mod 4). Benson and Lee (1989) showed a large class of cordial regular windmill graphs. Lee and Liu (1991) investigated cordial complete $k$-partite graphs. Kuo, Chang and Kwong (1997) determined all $m$ and $n$ for which $mK_n$ is cordial. Cordial generalized Petersen graphs are completely characterized by Ho, Lee and Shee (1989). In 1990, they investigated the construction of cordial graphs by Cartesian product and composition.

In this paper, we will exclusively focus on $A = Z_2$, and drop the reference to the group. In 2006, the friendly index set $FI(G)$ of a graph $G$ was introduced by Chartrand, Lee and Zhang. The set $FI(G)$ is defined as $\{|e_f(0) - e_f(1)| :$ the vertex labeling $f$ is friendly$\}$. When the context is clear, we will drop the subscript $f$.

Note that if 0 or 1 is in $FI(G)$, then $G$ is cordial. Thus the concept of friendly index sets could be viewed as a generalization of cordiality. Cairnie and Edwards (2000) have determined the computational complexity of cordial labeling and $Z_k$-cordial labeling. They proved that to decide whether a graph admits a cordial labeling is $NP$-complete. Even the restricted problem of deciding whether a connected graph of diameter 2 has a cordial labeling is $NP$-complete. Thus it is difficult to determine the friendly index sets of graphs.

In 2008, the friendly index sets of a few classes of graphs, in particular, complete bipartite graphs and cycles, are determined by Lee and Ng. It is shown that the star $K_{1,n}$ has friendly index set $FI(K_{1,n}) = \{1\}$ if $n$ is odd, and $\{0, 2\}$ if $n$ is even. The friendly index set of the corona of the path with $K_1$, $P_n \odot K_1$, is $\{1, 3, \ldots, 2n - 1\}$. For a full binary tree $T$ with depth 1, $FI(T) = \{0, 2\}$. A full binary tree $T$ with depth $d > 1$ has $FI(T) = \{0, 2, 4, \ldots, 2d + 1 - 4\}$. For more details of known results on friendly index sets, see Ho (2007), Kwong (2008), Lee and Ng (2007), Lee (2010), Salehi and Lee (2006).

**Theorem 1.1** For any graph with $q$ edges, the friendly index set $FI(G) \in \{0, 2, 4, \ldots, q\}$ if $q$ is even and $FI(G) \in \{1, 3, \ldots, q\}$ if $q$ is odd.

In 2008, Lee and Ng conjectured that the numbers in $FI$(T) for any tree T form an arithmetic progression. Various classes of tree were studied and their friendly index sets were shown to satisfy the conjecture. Counter-examples were eventually constructed by Salehi and De in 2008.

**Example 1**

The following tree with order 10 and size 9 has $FI(G) = \{1, 3, 5, 9\}$.

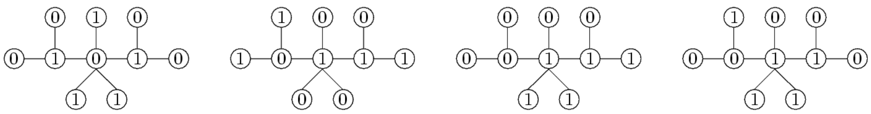


Figure 1: Tree with friendly indices not forming an arithmetic progression

Let $P_n$ be a path of $n$ vertices. A tree is called a spider if it has a center vertex $c$ with degree $k > 1$ while each of the other vertices is either a leaf or has degree 2. Thus, a spider is an amalgamation of $k$ paths with various lengths. If it has $x_1$ paths with length $a_1$, $x_2$ paths with length $a_2$, etc., we denote the spider by $SP(a_1^{x_1}, a_2^{x_2}, \ldots, a_m^{x_m})$, where $x_1 + x_2 + \ldots + x_m = k$ (see Figure 2.)

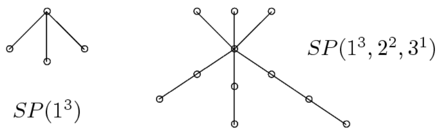



Figure 2: Examples of spider graphs

In this paper, we determined the friendly index sets of many spiders.

## 2. THE SPIDER $SP(1^n, a)$

The spider $SP(1^n, a)$ has $(n + a)$ edges and $(n + a + 1)$ vertices. For convenience, call the path from the center with length $a$ the stem, and the other $n$ paths with length 1 the branches. Let the vertices on the stem be sequentially $c, v_1, \ldots, v_a$, where $c$ is the center.

When we say label a branch, we mean label the vertex of the branch that is not the center. Without loss of generality, we may assume that the center has vertex label 0. Otherwise, change each vertex label $x$ to its complement $1 - x$, and the edge labels remain the same.

When $a = 0$ or 1, we have a star (see Lee and Ng (2008)). From now on, assume that $a \geq 2$.

**Theorem 2.1** Let $P_n$ denote the path with $n$ vertices. Then $FI(P_n) = \{1, 3, 5, \ldots, (n-1)\}$ if $n$ is even, and $= \{0, 2, 4, \ldots, (n-1)\}$ if $n$ is odd.

**Proof.** From Theorem 1.1, it suffices to show that all these integers are attainable. Label the vertices alternately with 0's and 1's, starting with 0. Note that the vertex labeling is friendly, with $v(1) - v(0) = 0$ when $n$ is even, and $v(1) - v(0) = -1$ when $n$ is odd.

First consider an even $n$. We make the following vertex label rearrangements. Keep the first and second vertex labels unchanged, and change all the remaining vertex labels on the path to their complements. Obviously $v(0)$ and $v(1)$ remain unchanged, and we have introduced exactly one zero edge label. Now keep the first four vertex labels (0, 1, 1, and 0) unchanged, and change all the remaining vertex labels on the path to their complements. Again, $v(0)$ and $v(1)$ remain unchanged, and we have introduced another zero edge label. Continue this to the end of the path, and we will have $(n/2) - 1$ edges labeled 0, and $n/2$ edges labeled 1. In this process, we have $e(0) = 0, 1, 2, \ldots, (n/2) - 1$. Thus $e(1) = n - 1, n - 2, n - 3, \ldots, n/2$, and so $e(1) - e(0) = n - 1, n - 3, n - 5, \ldots, 1$.

Now consider an odd $n$. We make exactly the same rearrangements as above. The vertex labelings remain friendly, with $v(1) - v(0)$ alternating between –1 and 1. The last step is to change the last vertex label to its complement, giving sequentially the edge labels 1, 0, 1, 0, . . . , 1, 0, and $e(1) = e(0)$.

**Theorem 2.2** $FI(\text{SP}(1^2, a)) = \{0, 2, 4, \ldots, a + 2\}$ if $a$ is even.

**Proof.** From Theorem 1.1, it suffices to show that all these integers are attainable. Note that the stem is the path $P_a + 1$, with an odd number of vertices. Label the center 0, both branches 1, and the vertices on the stem alternately. This labeling is friendly with $v(1) - v(0) = 1$, and all edge labels are 1s, giving $e(1) - e(0) = a + 2$. Then label the center 0, one branch 1 while the other branch 0, and the vertices on the stem alternately. This labeling is friendly with $v(1) - v(0) = -1$, and $e(1) - e(0) = a$. Use the procedure in the proof of Theorem 2.1 to rearrange the vertices on the stem to obtain $e(1) - e(0) = a - 2, a - 4, \ldots, 2$, and 0, with $v(1) - v(0)$ alternating between –1 and 1.

**Example 2**

$FI(SP(1^2, 4)) = \{0, 2, 4, 6\}$.

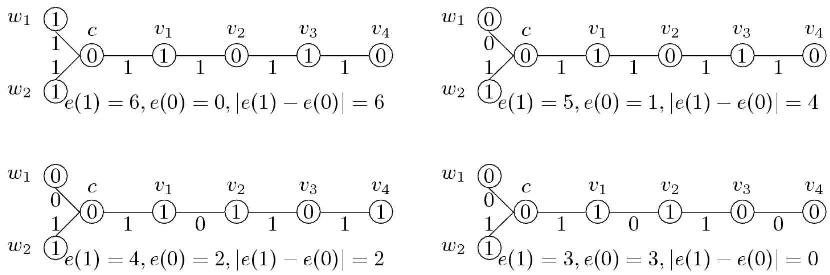



Figure 3: Friendly labelings for $SP(1^2, 4)$

**Theorem 2.3** $FI(SP(1^2, a) = \{1, 3, 5, \ldots, a\}$ if $a$ is odd.

**Proof.** We first show that $(a + 2) \notin FI(SP(1^2, a))$. Otherwise, $e(1) = 0$ or $e(0) = 0$. If $e(1) = 0$, all vertices have the same label. Such a labeling is not friendly. If $e(0) = 0$, all edges have label 1. Since the center has label 0, both branches have label 1, and the vertices on the stem starting from $v_1$ must have alternate labels 1 and 0. Since $a$ is odd, this labeling is not friendly.

Now we show that all positive odd integers up to $a$ are attainable. Label the center 0, the two branches by 0 and 1, and the vertices on the stem alternately. This labeling is friendly with $v(1) - v(0) = 0$, and $e(1) - e(0) = a$. Use the procedure in the proof of Theorem 2.1 to obtain all the other values.

**Example 3**

$FI(SP(1^2, 7)) = \{1, 3, 5, 7\}$.

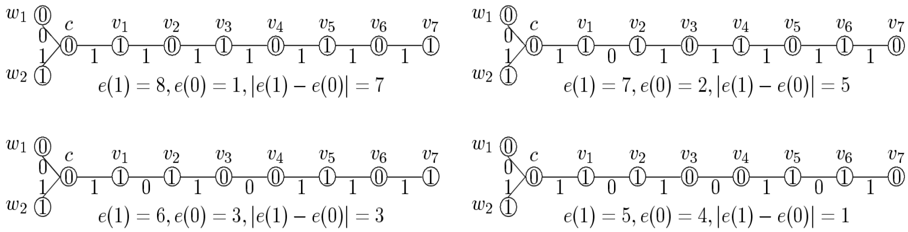


Figure 4: Friendly labelings for $SP(1^2, 7)$

Note that the cases $n = 0$, 1, and 2 have been settled in Theorems 2.1, 2.2 and 2.3. From now on, assume that $n \geq 3$.

**Lemma 2.1** $(n + a) \notin FI(SP(1^n, a))$, if $n \geq 3$ and $a \geq 2$.

**Proof.** Otherwise, $e(1) = 0$ or $e(0) = 0$. If $e(1) = 0$, all vertices have the same label. Such a labeling is not friendly. If $e(0) = 0$, all edges have label 1. Since the center has label 0, all branches (at least 3 of them) and $v_1$ must have label 1, and the other vertices on the stem have alternate labels. Such a labeling is not friendly.

We first consider $n < a$.

**Lemma 2.2** $\max\{FI(SP(1^n, a))\} = n + a - 2$, if $a > n \geq 3$.

**Proof.** It suffices to show that $|e(1) - e(0)| = n + a - 2$ is attainable.

Case 1: $(n + a)$ is odd, i.e., an odd number of edges and an even number of vertices. We label the vertices $v_{((a-n+1)/2)},\ldots,v_a$ on the stem with the label 1 and all other vertices of the graph with the label 0. Then $e(1) = 1$ and $e(0) = n + a - 1$, giving $|e(0) - e(1)| = n + a - 2$.

Case 2: $(n + a)$ is even, i.e., an even number of edges and an odd number of vertices. We label the vertices $v_{((a-n)/2)+1},\ldots,v_a$ on the stem with the label 1 and all other vertices of the graph with the label 0. Then $e(1) = 1$ and $e(0) = n + a - 1$, giving $|e(0) - e(1)| = n + a - 2$.

**Theorem 2.4** If $a > n \geq 3$, $FI(SP(1^n, a)) = \{n + a - 2, n + a - 4, \ldots\}$, ending at 1 if $(n + a)$ is odd, and ending at 0 if $(n + a)$ is even.

**Proof.** The labelings in Lemma 2.2 give $e(0) - e(1) = n + a - 2$.

Case 1: $(n + a)$ is odd.

Step 1: Interchange the vertex label 0 from one of the branches and the vertex label 1 at $v_{((a-n+1)/2)}$. Then $e(0)$ is decreased by 1 and $e(1)$ is increased by 1, giving $e(0) - e(1) = n + a - 4$. Continue to interchange vertex labels 0 from the branches and the vertex labels 1 at $v_{((a-n+1)/2)+1}, \ldots, v_{(a+n-1)/2}$. At that time, $e(0) = a - 1$ and $e(1) = n + 1$, giving $|e(0) - e(1)| = a - n - 2$.

Step 2: We now start another labeling as follows. Label all branches with 1, the vertices from $c$ to $v_{n-1}$ with 0, and the vertices from $v_n$ to $v_a$ (an even number of them) alternately with 0's and 1's. For this labeling, $e(1) = a$ and $e(0) = n$, and so $|e(1) - e(0)| = a - n$. For the path from $v_n$ to $v_a$, use the rearrangements in the proof of Theorem 2.1, keeping $|v(1) - v(0)|$ unchanged, and decreasing $|e(1) - e(0)|$ by 2 each time, until the difference becomes 1.

Case 2: $(n + a)$ is even.
The proof is exactly the same as above with these exceptions. In Step 1, start the interchange at vertex $v_{((a-n)/2)+1}$. In Step 2, the vertex $v_n$ has label 0, and the vertices from $v_{n+1}$ to $v_a$ (an even number of them) take alternate labels 0's and 1's. For this labeling, $|v(1) - v(0)| = 1$, and $e(1) = a - 1$, $e(0) = n + 1$. The rearrangements in the proof of Theorem 2.1 decrease $|e(1) - e(0)|$ by 2 each time, until the difference becomes 0.

## Example 4

$FI(SP(1^3, 4)) = \{1, 3, 5\}$.

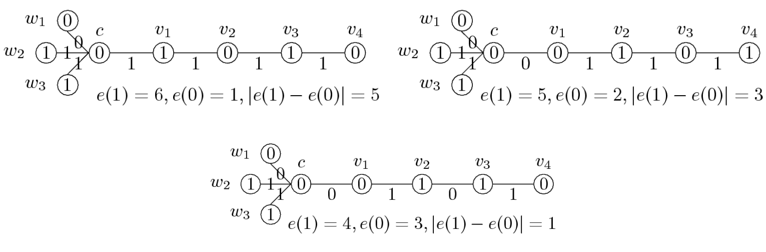



Figure 5: Friendly labelings for $SP(1^3, 4)$

**Example 5**

$FI(SP(1^4, 8) = \{0, 2, 4, 6, 8, 10\}$.

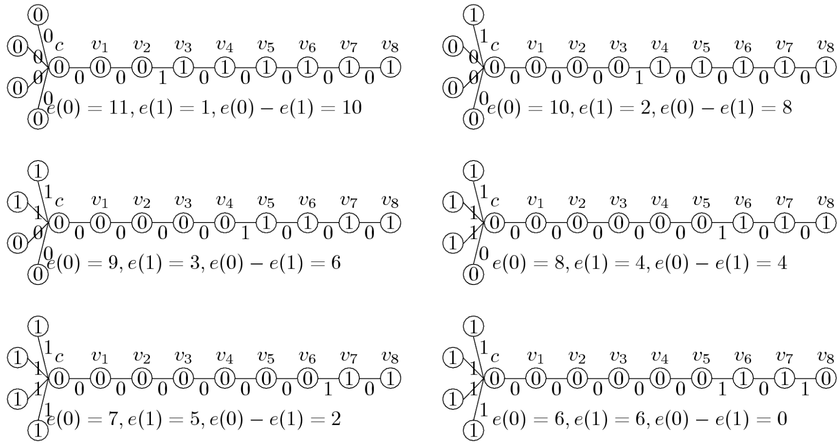


Figure 6: Friendly labelings for $SP(1^4, 8)$

Next consider $n \geq a$.

To find $\max\{FI(SP(1^n, a))\}$, we seek to maximize $e(0)$ or $e(1)$. If we put two complementary labels on alternate branches, they will give one 0-edge and one 1-edge, canceling each other in $|e(1) - e(0)|$. However if we label alternate vertices on the stem with complementary vertex labels, these alternate vertex labels will generate 1-edges. Thus to maximize $e(1)$, we should label the stem vertices alternately with 0's and 1's. On the other hand, a 1-vertex on each of $x$ branches will generate $x$ 1-edges. However, labeling $x$ vertices on the stem consecutively by 1's will generate $(x - 1)$ 0-edges. Thus to maximize $e(0)$, we should label the stem vertices consecutively by as many 1's as possible.

To maximize $e(0)$, consider two cases.

Case 1: $(n + a)$ is odd, i.e., an odd number of edges and an even number of vertices. To maximize $e(0)$, we label all vertices on the stem, other than the center, with the label 1. Then there must be $(n + a - 1)/2$ branches labeled 0, and $(n - a + 1)/2$ branches labeled 1. Thus $e(1) = (n - a + 1)/2 + 1$ and $e(0) = (n + a - 1)/2 + (a - 1)$, giving $|e(1) - e(0)| = 2a - 3$.

Case 2: $(n + a)$ is even, i.e., an even number of edges and an odd number of vertices. Again, we label all vertices on the stem, other than the center, with the label 1. Since $|v(1) - v(0)| = 1$, to maximize $e(0)$, there must be $(n + a)/2$ branches labeled 0, and $(n - a)/2$ branches labeled 1. Thus $e(1) = (n - a)/2 + 1$ and $e(0) = (n + a)/2 + (a - 1)$, giving $|e(1) - e(0)| = 2a - 2$.

To maximize $e(1)$, we put alternate complementary vertex labels on the stem to generate $a$ edge labels of 1.

Case 1: $n$ is even and $a$ is odd. On the stem, $v(1) - v(0) = 0$. Then there must be $n/2$ branches labeled 0, and $n/2$ branches labeled 1. Thus $e(1) = (n/2) + a$ and $e(0) = (n/2)$, giving $|e(1) - e(0)| = a$.

Case 2: $n$ is even and $a$ is even. On the stem, $v(1) - v(0) = -1$. To maximize $e(1)$, there must be $(n/2) - 1$ branches labeled 0, and $(n/2) + 1$ branches labeled 1. Thus $e(1) = (n/2) + 1 + a$ and $e(0) = (n/2) - 1$, giving $|e(1) - e(0)| = a + 2$.

Case 3: $n$ is odd and $a$ is odd. On the stem, $v(1) - v(0) = 0$. To maximize $e(1)$, there must be $(n - 1)/2$ branches labeled 0, and $(n + 1)/2$ branches labeled 1. Thus $e(1) = (n + 1)/2 + a$ and $e(0) = (n - 1)/2$, giving $|e(1) - e(0)| = a + 1$.

Case 4: $n$ is odd and $a$ is even. On the stem, $v(1) - v(0) = -1$. Then there must be $(n - 1)/2$ branches labeled 0, and $(n + 1)/2$ branches labeled 1. Thus $e(1) = (n + 1)/2 + a$ and $e(0) = (n - 1)/2$, giving $|e(1) - e(0)| = a + 1$.

**Theorem 2.5** $FI(SP(1^n, 2)) = \{1, 3\}$ if $n$ is odd, and $= \{0, 2, 4\}$ if $n$ is even.

**Proof.** By comparing the maximization of $e(0)$ and the maximization of $e(1)$, we see that $\max\{FI(SP(1^n, 2))\} = 3$ if $n$ is odd, and $= 4$ if $n$ is even.

First consider an odd $n$. Label $(n - 1)/2$ of the branches with 0, and the other $(n + 1)/2$ branches with 1. The vertices $c$, $v_1$, and $v_2$ are labeled with 0, 1, and 0 respectively. Then $v(0) = v(1)$, and $e(1) - e(0) = 3$. Then keep the branch labels the same, but label the vertices $c$, $v_1$, and $v_2$ with 0, 0, and 1 respectively. We have $v(0) = v(1)$, and $e(1) - e(0) = 1$.

Now consider an even $n$. Label $(n/2) - 1$ of the branches with 0, and the other $(n/2) + 1$ branches with 1. The vertices $c$, $v_1$, and $v_2$ are labeled with 0, 1, and 0 respectively. Then $v(1) - v(0) = 1$, and $e(1) - e(0) = 4$. Then keep the branch labels the same, but label the vertices $c$, $v_1$, and $v_2$ with 0, 0, and 1 respectively. We have $v(1) - v(0) = 1$, and $e(1) - e(0) = 2$. Finally, label half of the branches 0 and the remaining branches 1. Label the vertices $c$, $v_1$, and $v_2$ with 0, 0, and 1 respectively. Then $v(1) - v(0) = -1$, and $e(1) - e(0) = 0$.

**Lemma 2.3** For $n \geq a \geq 3$, $\max\{FI(SP(1^n, a))\} = 2a - 3$ if $(n + a)$ is odd, and $= 2a - 2$ if $(n + a)$ is even.

**Proof.** For $a = 3$ and $n$ odd, we have $2a - 2 \geq a + 1$. For $a = 3$ and $n$ even, we have $2a - 3 \geq a$. For $a \geq 4$, we have $2a - 3 \geq a + 1$, and $2a - 2 \geq a + 2$.

**Theorem 2.6** For $n \geq a \geq 3$, $FI(SP(1^n, a)) = \{2a - 3, 2a - 5, \dots, 1\}$ if $(n + a)$ is odd, and $= \{2a - 2, 2a - 4, \dots, 0\}$ if $(n + a)$ is even.

**Proof.** Consider the case when $(n + a)$ is even. From above, the value $(2a - 2)$ is attainable. Interchange the vertex label 0 from one of the branches and the vertex label 1 at $v_1$. Then $e(0)$ is decreased by 1 and $e(1)$ is increased by 1, giving $|e(1) - e(0)| = 2a - 4$. Continue to interchange vertex labels 0 from the branches and the vertex labels 1 at $v_2, \dots, v_{a-1}$. At that time, $e(0) = (n + a)/2 = e(1)$, giving $|e(1) - e(0)| = 0$.

Now consider the case when $(n + a)$ is odd. From above, the value $(2a - 3)$ is attainable. Interchange the vertex label 0 from one of the branches and the vertex label 1 at $v_1$. Then $e(0)$ is decreased by 1 and $e(1)$ is increased by 1, giving $|e(1) - e(0)| = 2a - 5$. Continue to interchange vertex labels 0 from the branches and the vertex labels 1 at $v_2, \dots, v_{a-2}$. At that time, $e(0) = (n + a + 1)/2$, and $e(1) = (n + a - 1)/2$, giving $|e(1) - e(0)| = 1$.

**Example 6**

$FI(SP(1^4, 3) = \{1, 3\}$.

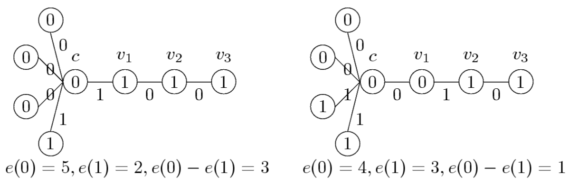



Figure 7: Friendly labelings for $SP(1^4, 3)$

**Example 7**

$FI(SP(1^5, 3)) = \{0, 2, 4\}$

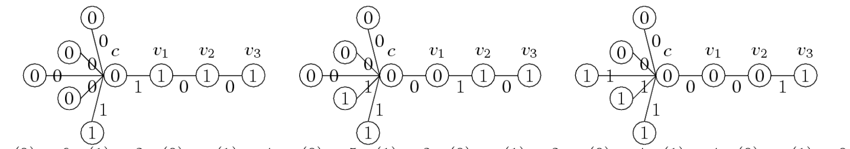



Figure 8: Frienldy labelings for $SP(1^5, 3)$

## 3. A GENERAL RESULT FOR SPIDERS

**Notation:** We let $q$ denote the number of edges. We will always start with the following labeling. Label the center vertex 0. Label all vertices with an odd distance from the center vertex 1, and all vertices with an even distance from the center 0. It is obvious that all edge labels are 1, and thus $e(1) - e(0) = q$. We will call this *the all-one labeling*. Note that this labeling is not necessarily friendly.

**Theorem 3.1** Let $a_1, a_2, \dots, a_m \geq 2$ be even. Then $FI(SP(a_1, a_2, \dots, a_m)) = \{0, 2, 4, \dots, q\}$.

**Proof.** From Theorem 1.1, it suffices to show that each even integer between 0 and q is attainable. We start with the all-one labeling. Note that the vertex labeling is friendly, with $v(1) - v(0) = -1$.

Consider each $i$ between 1 and $m$. For the path of the spider with $a_i$ edges, we make the following vertex label rearrangements. Keep the first (i.e., center) vertex label unchanged, and change all the remaining vertex labels on this path to their complements. Obviously $v(0)$ and $v(1)$ remain unchanged, and we have introduced exactly one zero edge label.

Now keep the first three vertex labels (0, 0, and 1) unchanged, and change all the remaining vertex labels on this path to their complements. Again, $v(0)$ and $v(1)$ remain unchanged, and we have introduced another zero edge label. Then keep the first five vertex labels (0, 0, 1, 1, and 0) unchanged and alter the remaining labels in the same way to introduce one more zero edge label while maintaining a friendly vertex labeling. Continue this to the end of the path, and we will have $(a_i/2)$ edges labeled 0, and $(a_i/2)$ edges labeled 1. Repeat this for all the remaining paths, until we have $(q/2)$ edges labeled 0, and $(q/2)$ edges labeled 1. In this process, we have $e(0) = 0, 1, 2,$

… , $(q/2)$. Thus $e(1) = q, q-1, q-2, \dots, (q/2)$, and so $e(1) - e(0) = q, q-2, q-4, \dots, 0$.

**Example 8**

The friendly index set $FI(SP(2^4)) = \{0, 2, 4, 6, 8\}$.

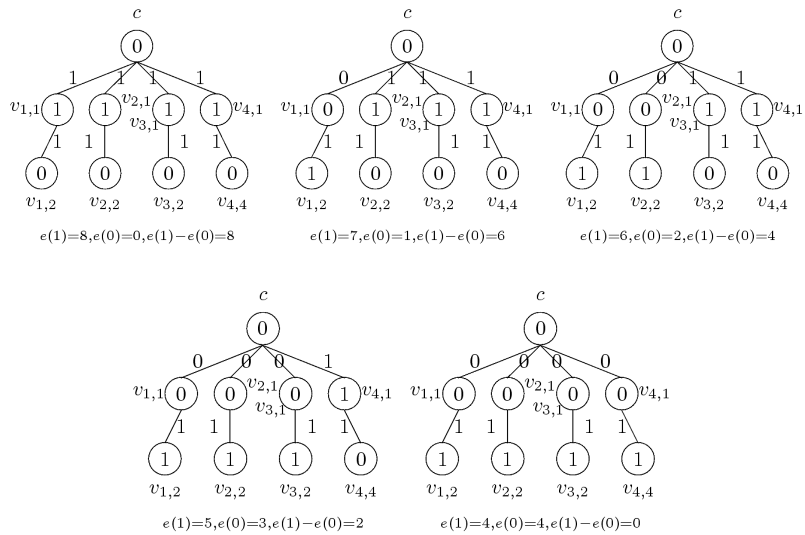



Figure 9: Friendly labelings for $SP(2^4)$

**Lemma 3.1** Let $b$ be odd. Then $FI(SP(b)) = \{1, 3, 5, \dots, b\}$.

**Proof.** Note that this spider is just a path. The result follows from Theorem 2.1.

**Lemma 3.2** Let $b_1$ and $b_2$ be odd. Then $FI(SP(b_1, b_2)) = \{0, 2, 4, \dots, q\}$.

**Proof.** Note that this spider is just a path of even edges. The result follows from Theorem 2.1.

**Lemma 3.3** Let $b_1$, $b_2$ and $b_3$ be odd. Then $FI(SP(b_1, b_2, b_3)) = \{1, 3, 5, \dots, q-2\}$.

**Proof.** Suppose $e(0) = 0$ for a given friendly labeling of $SP(b_1, b_2, b_3)$. Without loss of generality, we may assume the degree 3 vertex, $c$, is labeled with 0. Hence, the labeling must be the all-one labeling. However, this labeling has $v(1) - v(0) = 2$, a contradiction to the condition of friendly labeling. Obviously, $e(1) \neq 0$ too. Hence, $e(0)$, $e(1) \geq 1$ and $\max\{FI(SP(b_1, b_2, b_3))\} \leq q - 2$. Note that in the all-one labeling, all the end-vertices are labeled with 1. Now, change the vertex labels of $v_{3,1}$, $v_{3,2}$, …, $v_{3,b_3}$ to their complements, we have $v(1) = v(0)$ and $e(1) = 1$ with $\max\{FI(SP(b_1, b_2, b_3))\} = q - 2$ attained. From Theorem 1.1, it suffices to show that each odd integer between 1 and $q - 2$ is attainable.

Use Theorem 2.1 on the odd-length paths $P_{b_1+1}, P_{b_2+1}$ and $P_{b_3+1}$ respectively by keeping the vertex labels of c, $v_{1,1}$, $v_{2,1}$ and $v_{3,1}$ (0, 1, 1 and 0) unchanged. In this process, we produce labelings with $e(0) = 1, 2, 3, \ldots, (b_1 + b_2 + b_3 - 1)/2$. Thus, $e(1) = q - 1, q - 2, q - 3, \ldots, (b_1 + b_2 + b_3 + 1)/2$, and thus, $v(1) - v(0) = q - 2, q - 4, \ldots, 3, 1$.

**Lemma 3.4** Let $b_1$, $b_2$, $b_3$ and $b_4$ be odd. Then $FI(SP(b_1, b_2, b_3, b_4)) = \{0, 2, 4, \ldots, q - 2\}$.

**Proof.** Suppose $e(0) = 0$ for a given friendly labeling of $SP(b_1, b_2, b_3, b_4)$. Without loss of generality, we may assume the degree 4 vertex, $c$, is labeled with 0. Hence, the labeling must be the all-one labeling. However, this labeling has $v(1) - v(0) = 3$, a contradiction to the condition of friendly labeling. . Obviously, $e(1) \neq 0$ too. Hence, $e(0)$, $e(1) \geq 1$ and $\max\{FI(SP(b_1, b_2, b_3, b_4))\} \leq q - 2$. Note that in the all-one labeling, all the end-vertices are labeled with 1. Now, change the vertex labels of $v_{4,1}$, $v_{4,2}$, …, $v_{4,b_4}$ to their complements, we have $v(1) - v(0) = 1$ and $e(0) = 1$ with $\max\{FI(SP(b_1, b_2, b_3, b_4))\} = q - 2$ attained. From Theorem 1.1, it suffices to show that each even integer between 0 and $q - 2$ is attainable.

Suppose all of $b_1$, $b_2$, $b_3$ and $b_4 \geq 3$. Use Theorem 2.1 on the odd-length paths $P_{b_1+1}, P_{b_2+1}, P_{b_3+1}$ and $P_{b_4+1}$ respectively by keeping the vertex labels of $c$, $v_{1,1}$, $v_{2,1}$, $v_{3,1}$ and $v_{4,1}$ (0, 1, 1, 1 and 0) unchanged.
In this process, we produce labelings with $\max\{e(0)\} = (b_i - 1)/2$ for the paths $P_{b_i+1}$, $i = 1, 2, 3$, and $\max\{e(0)\} = (b_4 + 1)/2$ for the path $P_{b_4+1}$. Note that we always have $v(1) - v(0) = 1$. Now, if one of $b_i \equiv 1 \pmod 4$, then $v_{i,b_i}$ is labeled with 1, change the label of this vertex to 0. Otherwise, without loss of generality, change the last three vertex labels of the path $P_{b_1}$ (1, 1 and 0) to their complements. In this way, $v(0) - v(1) = 1$ and we have introduced one more zero edge label. Therefore, $e(0) = 1, 2, 3, \ldots, (b_1 + b_2 + b_3 + b_4)/2$. Thus, $e(1) = q - 1, q - 2, q - 3, \ldots, (b_1 + b_2 + b_3 + b_4)/2$, and $|v(1) - v(0)| = q - 2, q - 4, \ldots, 4, 2, 0$.

Note that at most three of the legs are paths of length one. Without loss of generality, we may assume $b_1 \geq 3$ and $b_4 = 1$. Use Theorem 2.1 on the odd path(s) of length $b_i \geq 3$ (respectively) by keeping the vertex labels of $c$, $v_{1,1}$, $v_{2,1}$, $v_{3,1}$ and $v_{4,1}$ (0, 1, 1, 1 and 0) unchanged. In this process, we produce labelings with $e(0) = 1, 2, 3, \ldots, (b_1 + b_2 + b_3 + b_4)/2 - 1$. Now, if $b_1 \equiv 1 \pmod 4$, then $v_{1,b_1}$ is labeled with 1, change the label of this vertex to 0. Otherwise, change the last three vertex labels of the path $P_{b_1}$ (1, 1 and 0) to their complements. Now, $v(0) - v(1) = 1$ and we have introduced one more zero edge label. Either way, we have $\max\{e(0)\} = (b_1 + b_2 + b_3 + b_4)/2$. Thus, $|v(1) - v(0)| = q - 2, q - 4, \ldots, 4, 2, 0$. €

**Lemma 3.5** Let $a_1, a_2, \ldots, a_m \geq 2$ be even and let $b$ be odd. Then $FI(SP(a_1, a_2, \ldots, a_m, b)) = \{1, 3, 5, \ldots, q\}$.

**Proof.** We start with the all-one labeling. Note that the vertex labeling is friendly, with $v(1) - v(0) = 0$.

Use Theorem 3.1 to produce labelings with $e(0) = 0, 1, \ldots, (a_1 + a_2 + \ldots + a_m)/2$. Note that $v(0)$ and $v(1)$ are both unchanged throughout this process.

Now use Lemma 3.1 to produce labelings with $e(0) = ((a_1 + a_2 + \ldots + a_m)/2) + 1, \ldots, (q - 1)/2$. Note that both $v(0)$ and $v(1)$ are unchanged throughout this process.

Note that $e(1) = q, q - 1, \ldots, (q + 1)/2$, and so $e(1) - e(0) = q, q - 2, \ldots, 1$.

**Lemma 3.6** Let $a_1, a_2, \ldots, a_m \geq 2$ be even and let $b_1$ and $b_2$ be odd. Then $FI(SP(a_1, a_2, \ldots, a_m, b_1, b_2)) = \{0, 2, 4, \ldots, q\}$.

**Proof.** We start with the all-one labeling. Note that the vertex labeling is friendly, with $v(1) - v(0) = 1$.

Use Theorem 3.1 to produce labelings with $e(0) = 0, 1, \ldots, (a_1 + a_2 + \ldots + a_m)/2$. Note that $v(0)$ and $v(1)$ are both unchanged throughout this process.

Now use Lemma 3.2 to produce labelings with $e(0) = (a_1 + a_2 + \ldots + a_m)/2 + 1, \ldots, q/2$. Note that $v(1) - v(0)$ changes from 1 to $-1$, and is friendly throughout the process.

Note that $e(1) = q, q-1, \dots, q/2$, and so $e(1) - e(0) = q, q-2, \dots, 0$.

**Lemma 3.7** Let $a_1, a_2, \dots, a_m \geq 2$ be even and let $b_1$, $b_2$ and $b_3$ be odd. Then $FI(SP(a_1, a_2, \dots, a_m, b_1, b_2, b_3)) = \{1, 3, 5, \dots, q-2\}$.

**Proof.** Suppose $e(0) = 0$ for a given friendly labeling of $SP(a_1, a_2, \dots, a_m, b_1, b_2, b_3)$. Without loss of generality, we may assume the degree $m$+3 vertex, $c$, is labeled with 0. Hence, the labeling must be the all-one labeling. However, this labeling has $v(1) - v(0) = 2$, a contradiction to the condition of friendly labeling. Hence, $e(0) \geq 1$ and $\max\{FI(SP(a_1, a_2, \dots, a_m, b_1, b_2, b_3))\} \leq q-2$. Note that in the all-one labeling, the last vertices of the three odd-length paths are labeled with 1. Now, change the vertex labels of $v_{3,1}$, $v_{3,2}$, …, $v_{3,b_3}$ to their complements, we have $v(1) = v(0)$ with $\max\{FI(SP(a_1, a_2, \dots, a_m, b_1, b_2, b_3))\} = q-2$ attained.

Use Theorem 3.1 to produce labelings with $e(0) = 1, 2, \dots, (a_1 + a_2 + \dots + a_m)/2 + 1$. Note that we still have $v(0) = v(1)$.

Now use Lemma 3.3 to produce labelings with $e(0) = (a_1 + a_2 + \dots + a_m)/2 + 2, \dots, (a_1 + a_2 + \dots + a_m)/2 + (b_1 + b_2 + b_3 - 1)/2$. Thus, $v(1) - v(0) = q-2, q-4, \dots, 3, 1$.

**Lemma 3.8** Let $a_1, a_2, \dots, a_m \geq 2$ be even and let $b_1$, $b_2$, $b_3$ and $b_4$ be odd. Then $FI(SP(a_1, a_2, \dots, a_m, b_1, b_2, b_3, b_4)) = \{0, 2, 4, \dots, q-2\}$.

**Proof.** Suppose $e(0) = 0$ for a given friendly labeling of $SP(a_1, a_2, \dots, a_m, b_1, b_2, b_3, b_4)$. Without loss of generality, we may assume the degree $m + 4$ vertex, $c$, is labeled with 0. Hence, the labeling must be the all-one labeling. However, this labeling has $v(1) - v(0) = 3$, a contradiction to the condition of friendly labeling.
Hence, $e(0) \geq 1$ and $\max\{FI(SP(a_1, a_2, \dots, a_m, b_1, b_2, b_3, b_4))\} \leq q-2$. Note that in the all-one labeling, the last vertices of the four odd-length paths are labeled with 1. Now, change the vertex labels of $v_{4,1}$, $v_{4,2}$, …, $v_{4,b_4}$ to their complements, we have $v(1) - v(0) = 1$ with $\max\{FI(SP(a_1, a_2, \dots, a_m, b_1, b_2, b_3, b_4))\} = q-2$ attained.

Use Theorem 3.1 to produce labelings with $e(0) = 1, 2, \dots, (a_1 + a_2 + \dots + a_m)/2 + 1$. Note that we still have $v(0) = v(1)$.

Now use Lemma 3.4 to produce labelings with $e(0) = (a_1 + a_2 + ... + a_m)/2 + 2, \ldots, (a_1 + a_2 + ... + a_m)/2 + (b_1 + b_2 + b_3 + b_4)/2$. Thus, $v(1) - v(0) = q - 2, q - 4, \ldots, 3, 1$.

**Theorem 3.2** Let $a_1, a_2, \ldots, a_m \geq 2$ be even and let $b_1, b_2, \ldots, b_n$ be odd. Then $FI(SP(a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n))$

(1) $= \{0, 2, 4, \ldots, q\}$ when $n = 0$ and 2,
(2) $= \{1, 3, 5, \ldots, q\}$ when $n = 1$,
(3) $= \{1, 3, 5, \ldots, q - 2\}$ when $n = 3$,
(4) $= \{0, 2, 4, \ldots, q - 2\}$ when $n = 4$,
(5) $\supseteq \{0, 2, 4, \ldots, q - n + 2\}$ when $n$ is even and $\geq 6$, and
(6) $\supseteq \{1, 3, 5, \ldots, q - n + 1\}$ when $n$ is odd and $\geq 5$.

**Proof.** When $n$ = 0, 1, 2, 3 and 4, the result follows from Theorem 3.1, Lemma 3.5, Lemma 3.6, Lemma 3.7 and Lemma 3.8, respectively.

Consider even $n \geq 6$. Start with the all-one labeling, which is not friendly, because $v(1) - v(0) = n - 1$. For each of the paths with $b_4, b_6, \ldots, b_n$ vertices, keep the center vertex label unchanged, and change the labels of all the other vertices to their complements. We have increased $v(0)$ by $(n - 2)/2$ and decreased $v(1)$ by $(n - 2)/2$. After this relabeling, $v(1) - v(0) = 1$, $e(0) = (n - 2)/2$, $e(1) = q - (n - 2)/2$, and $e(1) - e(0) = q - n + 2$. Use the procedure in Lemma 3.6 to relabel the vertices on the paths with $a_1, a_2, \ldots, a_m, b_1$, and $b_2$ edges. In this process, $e(0) = ((n - 2)/2) + 1, \ldots, ((n - 2)/2) + (a_1 + a_2 + \ldots + a_m + b_1 + b_2)/2$. For each of the other paths, i.e., those with $b_3, b_4, \ldots, b_n$ vertices, keep the first two vertex labels unchanged, and change the others to their complements. Then keep the first four vertex labels unchanged, and change the others to their complements, etc. The difference $v(1) - v(0)$ remains unchanged throughout this process. Eventually we will have $e(0) = q/2$, and so $e(1) - e(0) = 0$.

Consider odd $n \geq 5$. Start with the all-one labeling, which is not friendly, because $v(1) - v(0) = n - 1$. For each of the paths with $b_3, b_5, \ldots, b_n$ vertices, keep the center vertex label unchanged, and change the labels of all the other vertices to their complements. We have increased $v(0)$ by $(n - 1)/2$ and decreased $v(1)$ by $(n - 1)/2$. After this relabeling, $v(1) - v(0) = 0$, $e(0) = (n - 1)/2$, $e(1) = q - (n - 1)/2$, and $e(1) - e(0) = q - n + 1$. Use the procedure in Lemma 3.5 to relabel the vertices on the paths with $a_1, a_2, \ldots, a_m$, and $b_1$ edges.

In this process, $e(0) = ((n-1)/2) + 1, \ldots, ((n-1)/2) + (a_1 + a_2 + \ldots + a_m + b_1 - 1)/2$. For each of the other paths, i.e., those with $b_2, b_3, \ldots, b_n$ vertices, keep the first two vertex labels unchanged, and change the others to their complements. Then keep the first four vertex labels unchanged, and change the others to their complements, etc. The difference $v(1) - v(0)$ remains unchanged throughout this process. Eventually we will have $e(0) = (q - 1)/2$, and so $e(1) - e(0) = 1$.

**Corollary 3.1** $FI(SP(2^n, 2k+1)) = \{1, 3, 5, \ldots, 2(n+k)+1\}$ for any $n$, $k \geq 1$.

### Example 9

$FI(SP(1,2^2)=\{1, 3, 5\}$.

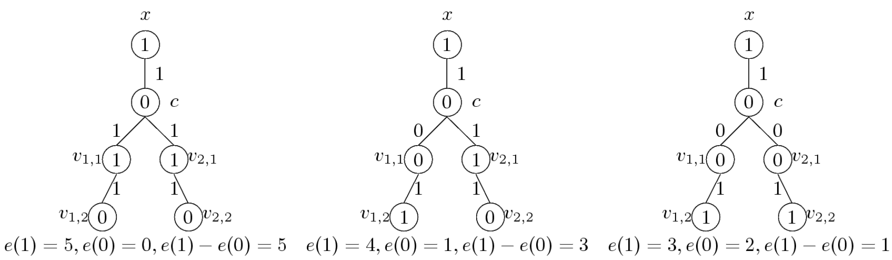
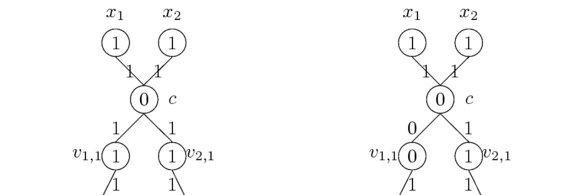



Figure 10: Friendly labelings for $SP(1, 2^2)$

### Example 10

$FI(SP(1^2, 2^2) = \{0, 2,4, 6\}$.



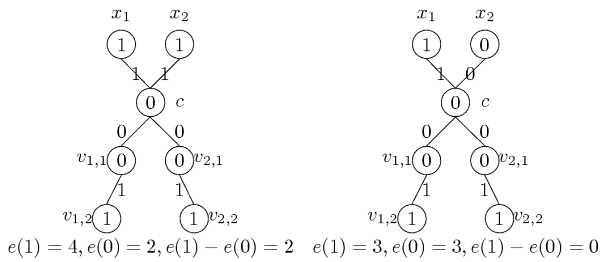



Figure 11: Friendly labelings for $SP(1^2, 2^2)$

## Example 11

$FI(SP(1^3, 2^2)) = \{1, 3, 5\}$

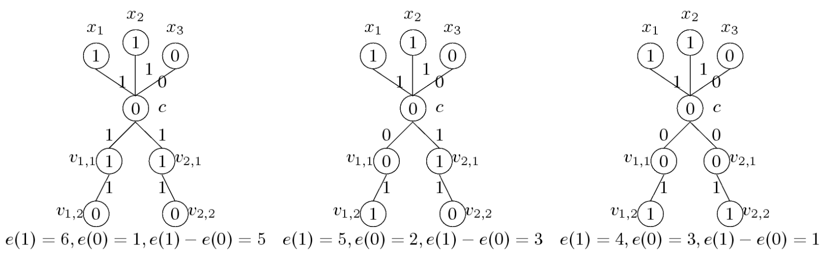
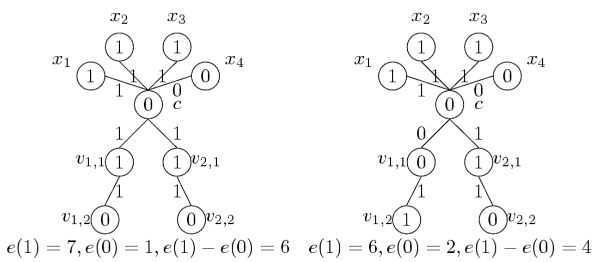



Figure 12: Friendly labelings for $SP(1^3, 2^2)$

## Example 12

$FI(SP(1^4, 2^2)) = \{0, 2, 4, 6\}$.



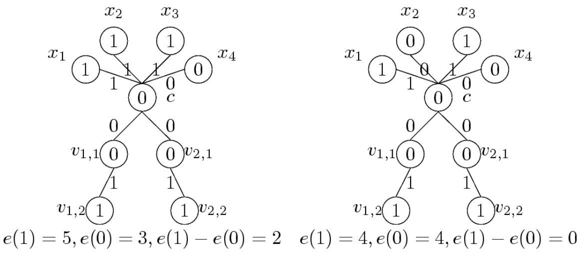



Figure 12: Friendly labelings for $SP(1^4, 2^2)$

## Example 13

$FI(SP(1^5, 2^2)) = \{1, 3, 5\}$.

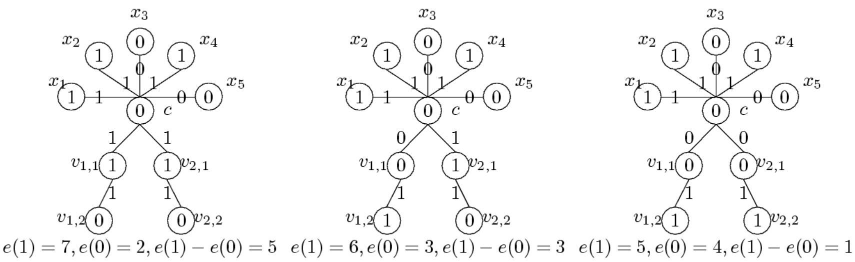
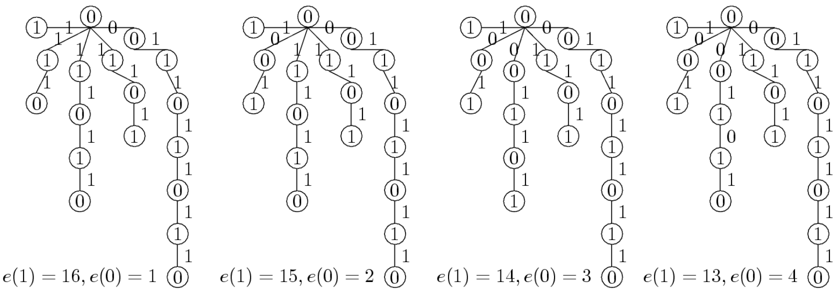



Figure 14: Friendly labelings for $SP(1^5, 2^2)$

## Example 14

$FI(SP(1,2,4,3,7)) = \{1, 3, 5, \ldots, 15\}$.



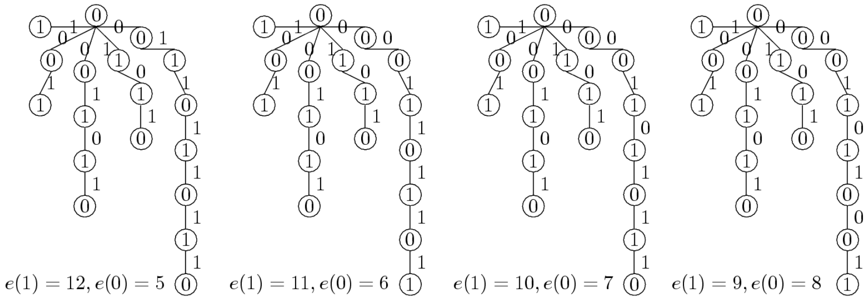



Figure 15: Friendly labelings for *SP*(1,2,4,3,7)

We end this paper by noting that the inclusions in Theorem 3.2 could be proper.

**Example 15**

Consider $SP(1^6, 6)$. By Theorem 2.6, $FI(SP(1^6, 6)) = \{0, 2, 4, 6, 8, 10\}$. However $q = 12$ and $n = 6$, and so $q - n + 2 = 8$.

## ACKNOWLEDGEMENTS

The authors gratefully acknowledge the valuable suggestions made by the referees.

## REFERENCES

Benson, M. and Lee, S-M. (1989). On cordialness of regular windmill graphs. *Congr. Numer*. **68**: 49-58.

Cahit, I. (1987). Cordial graphs: a weaker version of graceful and harmonious graphs. *Ars Combin*. **23**: 201-207.

Cahit, I. (1990). On cordial and 3-equitable graphs. *Util. Math*. **37**: 189-197.

Cahit, I. (1990). *Recent results and open problems on cordial graphs, in Contemporary Methods in Graph Theory*. 209-230, Bibligraphisches Inst., Mannhiem.

Cairnie, N. and Edwards, K. (2000). The computational complexity of cordial and equitable labelling. *Discrete Math*. **216**: 29-34.

Chartrand, G., Lee, S-M. and Zhang, P. (2006). On uniformly cordial graphs. *Discrete Math.* **306**: 726-737.

Ho, Y. S., Lee, S-M. and Shee, S.C. (1989). Cordial labellings of unicyclic graphs and generalized Petersen graphs. *Congr. Numer.* **68**: 109-122.

Ho, Y. S., Lee, S-M. and Shee, S. C. (1990). Cordial labellings of the Cartesian product and composition of graphs. *Ars Combin.* **29**: 169-180.

Ho, Y. S., Lee, S-M. and Ng, H. K. (2007). On friendly index sets of root-unions of stars by cycles. *J. Combin. Math. Combin. Comput.* **62**: 97-120.

Hovay, M. (1991). A-cordial graphs. *Discrete Math*. **93**: 183-194.

Kirchherr, W. W. (1991). On the cordiality of certain specific graphs. *Ars Combin.* **31**: 127-138.

Kirchherr, W. W. (1991). *Algebraic approaches to cordial labeling, Graph Theory, Combinatorics, Algorithms, and Applications*, Y. Alavi, *et. al.*, editors, SIAM, 294-299.

Kirchherr, W. W. (1993). NEPS operations on cordial graphs. *Discrete Math.* **115**: 201-209.

Kuo, S., Chang, G. J. and Kwong, Y. H. H. (1997). Cordial labeling of $mK_n$, *Discrete Math*. **169**: 121-131.

Kwong, H., Lee, S-M. and Ng, H. K. (2008). On friendly index sets of 2-regular graphs. *Discrete Math*. **308**: 5522-5532.

Lee, H. Y., Lee, H. M. and Chang, G. J. (1992). Cordial labelings of graphs. *Chinese J. Math*. **20**(3): 263-273.

Lee, S-M. and Liu, A. (1991). A construction of cordial graphs from smaller cordial graphs. *Ars Combin*. **32**: 209-214.

Lee, S-M. and Ng, H .K. (2008) On friendly index sets of bipartite graphs. *Ars Combin*. **86**: 257-271.

Lee, S-M. and Ng, H. K. (2007). On friendly index sets of total graphs of trees. *Util. Math*. **73**: 81-95.

Lee, S-M., Ng, H. K. and Tong, S. M. (2010). On friendly index sets of broken wheels with three spokes. *J. Combin. Math. Combin. Comput*. **74**: 13-31.

Salehi, E. and De, S. (2008). On a conjecture concerning the friendly index set of trees. *Ars Combin*. **90**: 371-381.

Salehi, E. and Lee, S-M. (2006). On friendly index sets of trees. *Congr. Numer*. **178**: 173-183.

Seah, E. (1991). On the construction of cordial graphs. *Ars Comb.* **31**: 249-254.

Seoud, M. A. and Abdel Maqsoud, A. E. I. (1999). On cordial and balanced labelings of graphs. *J. Egyptian Math. Soc.* **7**: 127-135.

Shee, S. C. and Ho, Y. S. (1996). The cordiality of the path-union of n copies of a graph. *Discrete Math*. **151**: 221-229.

Shee, S. C. and Ho, Y. S. (1993). The cordiality of one-point union of n copies of a graph. *Discrete Math.* **117**: 225-243.